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Poisson structures transverse to coadjoint orbits

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Abstract

We show that the Poisson structure transverse to a coadjoint orbit in the dual of a semisimple Lie algebra has a polynomial structure matrix, as conjectured by Damianou.
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Résumé

On démontre que la structure de Poisson transverse à une orbite coadjointe dans le dual d'un algèbre de Lie semisimple a une matrice de structure polynomiale, comme il a été conjecturé par Damianou.
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Introduction

Let M be a finite dimensional Poisson manifold, S the symplectic leaf through a point μ in M, and N a submanifold of M containing μ which satisfies $T_\mu N \oplus T_\mu S = T_\mu M$. Weinstein proved that there exists a Poisson structure on a neighbourhood U of μ in N such that a neighbourhood of μ in M is isomorphic as a Poisson manifold to the product of U with a neighbourhood of μ in S [11]. The isomorphism class of the transverse Poisson

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structure to S is independent of the choice of point μ in S and section N through μ. Examples of these transverse structures have been computed for the case when M is the dual of a Lie algebra, $\mathfrak{g}^*$ say, with its standard Poisson structure, for which the symplectic leaves are coadjoint orbits of the associated connected Lie group. In this case the section N can be chosen to be an affine subspace of $\mathfrak{g}^*$ and then the entries in the structure matrix of the transverse Poisson structure will be rational functions of coordinates on N. Computations show that for nilpotent Lie algebras [9] and semidirect products (Section 2 below) the entries need not be polynomial. However in [2] it is reported that an extensive series of computations for $\mathfrak{g} = \mathrm{sl}_n(\mathbb{C})$ showed that in all cases considered the transverse Poisson structure *is* polynomial. It was therefore natural to conjecture that this is always the case for $\mathrm{sl}_n(\mathbb{C})$. In this paper we prove that the transverse Poisson structure to a coadjoint orbit of *any* semisimple Lie algebra has a polynomial structure matrix.

In Section 1 we briefly recall the local block diagonalisation of the structure matrix of an arbitrary Poisson manifold given by Weinstein. In Section 2 we describe how to compute the Poisson structure transverse to a coadjoint orbit of $M = \mathfrak{g}^*$. Finally, in Section 3 we use the Jacobson–Morosov theorem and $\mathrm{sl}_2(\mathbb{C})$ representation theory to prove the main result.

1. Transverse Poisson structures

In this section we recall the notion of transverse Poisson structures using Weinstein's local splitting theorem [11, pp. 529–531]. Given the Poisson tensor Λ and a transversal N to a symplectic leaf S of the Poisson manifold (M, Λ), we present a decomposition of T^*M in which Λ is block diagonal.

Let M be a Poisson manifold with Poisson structure $\Lambda : T^*M \oplus T^*M \to \mathbb{R}$ which induces the linear map $\Lambda^\sharp : T^*M \to TM$ defined by $\Lambda^\sharp_\mu(\xi) = (\Lambda_\mu)_\xi$. For $\mu \in M$, let S be the symplectic leaf through μ. Then $T_\mu S = \operatorname{im} \Lambda^\sharp_\mu$. Let N be a submanifold of M containing the point μ such that $T_\mu N \oplus T_\mu S = T_\mu M$. We use the superscript $\circ$ to denote the annihilator of a subspace of a vector space.

Lemma 1.1. *There exists a neighbourhood U of μ in N such that*

$$T_U M = T_U N \oplus \Lambda^\sharp\big((T_U N)^\circ\big). \tag{1}$$

Proof. At $\mu \in S$ the vector bundle map $\Lambda^\sharp$ has kernel $(T_\mu S)^\circ$ and so maps $(T_\mu N)^\circ$ bijectively onto $T_\mu S$. Hence

$$T_\mu M = T_\mu N \oplus \Lambda^\sharp\big((T_\mu N)^\circ\big).$$

Moreover, the restriction of $\Lambda^\sharp$ to the subvector bundle $(TN)^\circ$ has maximum possible rank at μ. So there exists a neighbourhood U of μ in N on which the rank of $\Lambda^\sharp|_{(TN)^\circ}$ is constant. Consequently,

$$T_\nu M = T_\nu N \oplus \Lambda^\sharp\big((T_\nu N)^\circ\big),$$

for every $\nu \in U$. $\square$

The decomposition of T^*M dual to the decomposition (1) is given by

$$T_U^*M = (T_U N)^\circ \oplus \big(\Lambda^\sharp\big((T_U N)^\circ\big)\big)^\circ. \tag{2}$$

Let π_S and π_N denote the projections onto the first and second summand of decomposition (2), respectively.

Corollary 1.2. *If $\xi_i \in T_U^*M$ for $i = 1, 2$ then*

$$\Lambda(\xi_1, \xi_2) = \Lambda\big(\pi_S(\xi_1), \pi_S(\xi_2)\big) + \Lambda\big(\pi_N(\xi_1), \pi_N(\xi_2)\big).$$

Proof. This follows from

$$\Lambda\big(\pi_S(\xi_1), \pi_N(\xi_2)\big) = \Lambda^\sharp\big(\pi_S(\xi_1)\big)\big(\pi_N(\xi_2)\big) = 0,$$

since $\pi_S(\xi_1) \in (T_U N)^\circ$ and $\pi_N(\xi_2) \in (\Lambda^\sharp((T_U N)^\circ))^\circ$. □

The projection $\pi_N : T_U^*M \to (\Lambda^\sharp((T_U N)^\circ))^\circ$ can be identified with $(Ti)^* : T^*M \to T^*N$, where $i : N \hookrightarrow M$ is the inclusion of N into M. Hence $(\Lambda^\sharp((T_U N)^\circ))^\circ$ can be identified with T_U^*N. Thus we can make $U \subseteq N$ into a Poisson manifold by defining

$$\Lambda_U(\xi_1, \xi_2) = \Lambda(\hat{\xi}_1, \hat{\xi}_2),$$

where $\hat{\xi}_i$ is the inverse image of $\xi_i \in T^*N$ in $(\Lambda^\sharp((T_U N)^\circ))^\circ$ under the map $(Ti)^*$. We call this the *transverse Poisson structure* to S. The Poisson bracket of two smooth functions on N satisfies the Jacobi identity because the symplectic leaves of the Poisson manifold M intersect N in symplectic manifolds [11].

In [11] Weinstein shows that different choices of $\mu \in S$ and different choices of submanifolds N give isomorphic transverse Poisson structures. Moreover, he shows that a neighbourhood of $\mu \in M$ is isomorphic as a Poisson manifold to the product of a neighbourhood of μ in S and the transverse Poisson structure to S.

2. Coadjoint actions

We now specialize the above discussion to the case when $M = \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual space of a finite dimensional Lie algebra $\mathfrak{g}$ with its Poisson structure

$$\Lambda_\mu(\xi_1, \xi_2) = \mu([\xi_1, \xi_2]) = \mathrm{ad}^*_{\xi_1}(\mu)(\xi_2).$$

Here $\xi_i \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. The symplectic leaves of this Poisson manifold are the coadjoint orbits on $\mathfrak{g}^*$ of the connected Lie group G whose Lie algebra is $\mathfrak{g}$. Let $\mathfrak{g}_\mu$ be the isotropy subalgebra (i.e., the Lie algebra of the isotropy group G_μ) of the coadjoint action at μ. Let $\mathfrak{n}_\mu$ be a complement to $\mathfrak{g}_\mu$ in $\mathfrak{g}$, so $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{n}_\mu$ and $\mathfrak{g}^* = \mathfrak{g}_\mu^\circ \oplus \mathfrak{n}_\mu^\circ$. Note that this latter decomposition induces an isomorphism between $\mathfrak{n}_\mu^\circ$ and $\mathfrak{g}_\mu^*$.

Lemma 2.1. *The affine subspace $N = \mu + \mathfrak{n}_\mu^\circ$ is transverse at μ to the coadjoint orbit $\mathrm{Ad}^*_G(\mu)$ through μ.*

Proof. We have

$$T_\mu \mathrm{Ad}^*_G(\mu) = \mathrm{ad}^*_{\mathfrak{g}}(\mu) = \mathfrak{g}^\circ_\mu$$

and so $T_\mu N = \mathfrak{n}^\circ_\mu$ is a complement to $T_\mu \mathrm{Ad}^*_G(\mu)$ in $(T_\mu \mathfrak{g})^* = \mathfrak{g}^*$. □

If $\nu \in \mathfrak{n}^\circ_\mu$ then we have the identifications

$$T_{\mu+\nu} N \simeq \mathfrak{n}^\circ_\mu,$$
$$(T_{\mu+\nu} N)^\circ \simeq \mathfrak{n}_\mu$$

and

$$\Lambda^\sharp\big((T_{\mu+\nu}N)^\circ\big) \simeq \mathrm{ad}^*_{\mathfrak{n}_\mu}(\mu+\nu).$$

Moreover,

$$\begin{aligned}\big(\Lambda^\sharp\big((T_{\mu+\nu}N)^\circ\big)\big)^\circ &\simeq \big\{\xi \in \mathfrak{g} \mid \mathrm{ad}^*_{\mathfrak{n}_\mu}(\mu+\nu)(\xi) = 0\big\}\\ &= \big\{\xi \in \mathfrak{g} \mid \mathrm{P}_{\mathfrak{g}^\circ_\mu}(\mathrm{ad}^*_\xi(\mu+\nu)) = 0\big\},\end{aligned}$$

where $\mathrm{P}_{\mathfrak{g}^\circ_\mu}$ is the projection $\mathfrak{g}^* \to \mathfrak{g}^\circ_\mu$ with kernel $\mathfrak{n}^\circ_\mu$.

Lemma 2.2 [8]. *There exists a neighbourhood U of 0 in $\mathfrak{n}^\circ_\mu$ such that for every $\nu \in U$ and every $\xi \in \mathfrak{g}_\mu$ the equation*

$$\mathrm{P}_{\mathfrak{g}^\circ_\mu}\big(\big(\mathrm{ad}^*_{\xi+\eta}(\mu+\nu)\big)\big) = 0 \tag{3}$$

has a unique solution $\eta = \eta_{\mu+\nu}(\xi) \in \mathfrak{n}_\mu$. The map $\eta : (\mu+U) \oplus \mathfrak{g}_\mu \to \mathfrak{n}_\mu$ is smooth in ν, linear in ξ and vanishes identically when $\nu = 0$.

Proof. Write (3) as an inhomogeneous linear equation for $\eta \in \mathfrak{n}_\mu$, namely,

$$\mathrm{P}_{\mathfrak{g}^\circ_\mu}\big(\mathrm{ad}^*_\eta(\mu+\nu)\big) = -\mathrm{P}_{\mathfrak{g}^\circ_\mu}\big(\mathrm{ad}^*_\xi(\mu+\nu)\big) = -\mathrm{P}_{\mathfrak{g}^\circ_\mu}\big(\mathrm{ad}^*_\xi(\nu)\big). \tag{4}$$

Eq. (4) has a unique solution $\eta = \eta_{\mu+\nu}(\xi)$ if and only if the linear operator $L(\nu)(\eta) = \mathrm{P}_{\mathfrak{g}^\circ_\mu}(\mathrm{ad}^*_\eta(\mu+\nu))$ is invertible. When $\nu = 0$, we have

$$\begin{aligned}\mathrm{P}_{\mathfrak{g}^\circ_\mu}\big(\mathrm{ad}^*_\eta(\mu)\big) = 0 &\iff \mathrm{ad}^*_\eta(\mu)(\mathfrak{n}_\mu) = 0\\ &\iff \mathrm{ad}^*_{\mathfrak{n}_\mu}(\mu)(\eta) = 0\\ &\iff \eta \in \mathfrak{g}_\mu,\end{aligned}$$

since

$$\big(\mathrm{ad}^*_{\mathfrak{n}_\mu}(\mu)\big)^\circ = \big(\mathrm{ad}^*_{\mathfrak{g}}(\mu)\big)^\circ = \mathfrak{g}_\mu.$$

Consequently, $\eta \in \mathfrak{g}_\mu \cap \mathfrak{n}_\mu = \{0\}$. Hence at $\nu = 0$ the operator L is injective and thus is invertible. By continuity, L remains invertible on some neighbourhood U of 0. □

Corollary 2.3. *The Poisson structure transverse at μ to the coadjoint orbit through $\mu \in \mathfrak{g}^*$ is isomorphic to the Poisson structure in a neighbourhood of 0 in $\mathfrak{n}^\circ_\mu \simeq \mathfrak{g}^*_\mu$ defined by*

$$\Lambda_{N,\mu+\nu}(\xi_1,\xi_2) = (\mu+\nu)\big([\xi_1 + \eta_{\mu+\nu}(\xi_1), \xi_2 + \eta_{\mu+\nu}(\xi_2)]_{\mathfrak{g}}\big), \tag{5}$$

for every $\xi_i \in \mathfrak{g}_\mu$, or equivalently

$$\Lambda_{N,\mu+\nu}(\xi_1,\xi_2) = \nu\big([\xi_1,\xi_2]_{\mathfrak{g}_\mu}\big) - (\mu+\nu)\big([\eta_{\mu+\nu}(\xi_1),\eta_{\mu+\nu}(\xi_2)]_{\mathfrak{g}}\big), \tag{6}$$

where the subscripts $\mathfrak{g}$ and $\mathfrak{g}_\mu$ denote the Lie bracket on $\mathfrak{g}$ and $\mathfrak{g}_\mu$, respectively.

Proof. By Eq. (2) and Lemma 2.2 we can identify $(\Lambda^\sharp((T_{\mu+\nu}N)^\circ))^\circ$ with the subspace $\{\xi + \eta_{\mu+\nu}(\xi) \mid \xi \in \mathfrak{g}_\mu\}$ contained in $T^*_{\mu+\nu}M$. Eq. (5) then follows from the definition of the transverse Poisson structure on N. To obtain the equivalent form (6) note that

$$\mu\big([\xi_1,\xi_2]_{\mathfrak{g}_\mu}\big) = \mathrm{ad}^*_{\xi_1}(\mu)(\xi_2) = 0,$$

since $\mathrm{ad}^*_\xi(\mu) = 0$ for all $\xi \in \mathfrak{g}_\mu$, and

$$(\mu+\nu)\big([\xi_1 + \eta_{\mu+\nu}(\xi_1),\eta_{\mu+\nu}(\xi_2)]\big) = \mathrm{ad}^*_{\xi_1+\eta_{\mu+\nu}(\xi_1)}(\mu+\nu)\big(\eta_{\mu+\nu}(\xi_2)\big) = 0,$$

by the definition of $\eta_{\mu+\nu}(\xi)$ in Lemma 2.2. □

Remark 2.4. Note that the first term on the right hand side of (6) is just the standard Lie–Poisson structure on $\mathfrak{g}^*_\mu$. Since $\eta_\mu = 0$, the Taylor series in ν of the term $\mu([\eta_{\mu+\nu}(\xi_1),\eta_{\mu+\nu}(\xi_2)]_{\mathfrak{g}})$ starts with terms of degree 2 and the term $\nu([\eta_{\mu+\nu}(\xi_1), \eta_{\mu+\nu}(\xi_2)]_{\mathfrak{g}})$ with terms of degree 3. It follows that the linearization of the transverse Poisson structure at μ is the standard Lie–Poisson structure on $\mathfrak{g}^*_\mu$. A number of special cases are known for which some or all of the higher order terms vanish:

(1) If μ is a regular element of $\mathfrak{g}^*$ then the transverse Poisson structure is trivial and $\mathfrak{g}_\mu$ is abelian, see [3,11].
(2) If there exists a complementary subspace $\mathfrak{n}_\mu$ to $\mathfrak{g}_\mu$ in $\mathfrak{g}$ such that $[\mathfrak{g}_\mu,\mathfrak{n}_\mu] \subseteq \mathfrak{n}_\mu$, i.e., μ is *split* [5,8], then $\eta_{\mu+\nu} \equiv 0$ and the transverse Poisson structure is isomorphic to the standard Lie–Poisson structure on $\mathfrak{g}^*_\mu$, see [11].
(3) If there is a complementary subspace $\mathfrak{n}_\mu$ to $\mathfrak{g}_\mu$ in $\mathfrak{g}$ such that $[\mathfrak{n}_\mu,\mathfrak{n}_\mu] \subseteq \mathfrak{n}_\mu$, i.e., μ is *conormal* [8], then

$$\Lambda_{N,\mu+\nu}(\xi_1,\xi_2) = \nu\big([\xi_1,\xi_2]_{\mathfrak{g}_\mu}\big) + \mu\big([\eta_{\mu+\nu}(\xi_1),\eta_{\mu+\nu}(\xi_2)]_{\mathfrak{n}_\mu}\big). \tag{7}$$

Moreover, $\eta_{\mu+\nu}$ is linear in ν, and so the transverse Poisson structure is at most quadratic in ν, see [7].

The expressions (5) and (6) for the transverse Poisson structure reduce its computation to determining the map

$$\eta : (\mu + U) \times \mathfrak{g}_\mu \to \mathfrak{n}_\mu.$$

To do this we have to solve Eq. (3), or equivalently:

$$\mathrm{P}_{\mathfrak{g}^\circ_\mu}\big(\mathrm{ad}^*_\eta(\mu)\big) + \mathrm{P}_{\mathfrak{g}^\circ_\mu}\big(\mathrm{ad}^*_\eta(\nu)\big) = -\mathrm{P}_{\mathfrak{g}^\circ_\mu}\big(\mathrm{ad}^*_\xi(\nu)\big). \tag{8}$$

Example. Let $\mathfrak{g} = \mathrm{se}(3)$, the Lie algebra of the special Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$. Here $\ltimes$ denotes a semidirect product. Identify $\mathfrak{g}$ with $\mathbb{R}^3 \oplus \mathbb{R}^3$, where the first $\mathbb{R}^3$ is $\mathrm{so}(3)$. Use the standard inner product on $\mathbb{R}^3$ to identify $\mathfrak{g}^*$ with $\mathbb{R}^3 \oplus \mathbb{R}^3$. Of course this identification does not identify the adjoint action with the coadjoint action of SE(3). With respect to the above identifications we have

$$\begin{aligned}
&\mathrm{ad}_\xi(\eta) = (\mathbf{a}\times\mathbf{c}, \mathbf{a}\times\mathbf{d} - \mathbf{c}\times\mathbf{b}),\\
&\mathrm{ad}^*_\xi(\mu) = -(\mathbf{a}\times\mathbf{x} + \mathbf{b}\times\mathbf{y}, \mathbf{a}\times\mathbf{y}),
\end{aligned}$$

where $\xi = (\mathbf{a}, \mathbf{b})$, $\eta = (\mathbf{c}, \mathbf{d})$, and $\mu = (\mathbf{x}, \mathbf{y})$.

The proper isotropy subgroups for the coadjoint action of SE(3) are conjugate to either $\mathrm{SO}(2)\times\mathbb{R}$ or $\mathrm{SO}(2)\ltimes\mathbb{R}^3$. The points with isotropy subgroups conjugate to $\mathrm{SO}(2)\times\mathbb{R}$ are regular and so have trivial transverse Poisson structures. The points with isotropy subgroup conjugate to $\mathrm{SO}(2)\ltimes\mathbb{R}^3$ are of the form $\mu = (\mathbf{x}^\circ, 0)$ where $\mathbf{x}^\circ = (0, 0, x_3^\circ)$. For these points we have $\mathfrak{g}_\mu = \mathrm{so}(2)\oplus\mathbb{R}^3$, where

$$\mathrm{so}(2) = \left\{\mathbf{x}\in\mathbb{R}^3 \mid \mathbf{x} = (0, 0, x_3)\right\}.$$

Let $\eta_\mu = \mathrm{so}(2)^\perp\oplus\{0\}$. Here $\perp$ denotes the orthogonal complement with respect to the standard inner product. The identification $\mathfrak{g}^* = \mathbb{R}^3\oplus\mathbb{R}^3$ induces the identification of $\mathfrak{g}^\circ_\mu$ with $\mathfrak{n}_\mu$ and $\mathfrak{n}^\circ_\mu$ with $\mathfrak{g}_\mu$. Let

$$\begin{aligned}
\nu &= \big((0, 0, x_3), (y_1, y_2, y_3)\big)\in\mathfrak{n}^\circ_\mu,\\
\xi &= \big((0, 0, a_3), (b_1, b_2, b_3)\big)\in\mathfrak{g}_\mu,\\
\eta &= \big((A_1, A_2, 0), (0, 0, 0)\big)\in\mathfrak{n}_\mu.
\end{aligned}$$

Then Eq. (8) is

$$\left(x_3^\circ + x_3\right)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix} = \begin{pmatrix}b_2y_3 - b_3y_2\\ b_3y_1 - b_1y_3\end{pmatrix}.$$

Inverting the left hand side of the above equation gives

$$\eta_{\mu+\nu}(\xi) = \frac{1}{(x_3^\circ + x_3)}\begin{pmatrix}b_3y_1 - b_1y_3\\ b_3y_2 - b_2y_3\end{pmatrix}.$$

Substituting the above result into Eq. (6) gives the following expression for the transverse Poisson structure (considered as a 4×4 skew-symmetric matrix)

$$\Lambda_{N,(\mathbf{x},\mathbf{y})} = \left(\begin{array}{r|ccc} 0 & y_2 & -y_1 & 0\\ \hline -y_2 & & & \\ y_1 & & A & \\ 0 & & & \end{array}\right),$$

where

$$A = \frac{y_3}{x_3^\circ + x_3}\begin{pmatrix}0 & -y_3 & y_2\\ y_3 & 0 & -y_1\\ -y_2 & y_1 & 0\end{pmatrix}.$$

Remark 2.5. In the above example the Poisson structure Λ_N is a nontrivial rational function of the coordinates on the transverse section $\mu + \mathfrak{n}^\circ_\mu$. Calculations of Poisson structures transverse to coadjoints orbits of nilpotent Lie algebras [9] show that these are also typically rational. On the other hand, calculations for $\mathrm{gl}(n, \mathbb{C})$ led to the conjecture that Poisson structures transverse to coadjoint orbits of semisimple Lie algebras *are polynomial* [2]. Eq. (8) suggests a method for proving this. The equations can be written in the form

$$L(\nu)\eta = E\nu, \tag{9}$$

where $L(\nu) = D + F\nu$ and

$$D : \mathfrak{n}_\mu \to \mathfrak{g}_\mu^\circ, \quad E : \mathfrak{n}_\mu^\circ \to \mathfrak{g}_\mu^\circ, \quad \text{and} \quad F : \mathfrak{n}_\mu^\circ \to \mathrm{Hom}\big(\mathfrak{n}_\mu, \mathfrak{g}_\mu^\circ\big)$$

are linear operators. The proof of Lemma 2.2 shows that D is invertible. If for each ν the operator $D^{-1}F\nu$ is nilpotent of index r, say, then $L(\nu)^{-1}$ will be a polynomial of degree at most r in ν. Hence the map $\eta_{\mu+\nu}$ will be a polynomial of degree at most $r + 1$ and the transverse Poisson structure will be polynomial of degree at most $2(r + 1) + 1$. In the next section we show that this is indeed the case for semisimple Lie algebras.

Remark 2.6. Dirac brackets can also be used to calculate the matrix of the transverse Poisson structure with respect to a specific basis for $T^*_{\mu+\nu}N \cong \mathfrak{g}_\mu$ (see, for example, [9]). Let $\{\xi_i\}_{i=1}^k$ and $\{\zeta_i\}_{i=k+1}^n$ be bases for $\mathfrak{g}_\mu$ and $\mathfrak{n}_\mu$, respectively. For $\nu \in \mathfrak{n}_\mu^\circ$ define matrices A, B, and C by

$$A_{ij}(\nu) = (\mu + \nu)([\xi_i, \xi_j]),$$
$$B_{ij}(\nu) = (\mu + \nu)([\xi_i, \zeta_j]),$$
$$C_{ij}(\nu) = (\mu + \nu)([\zeta_i, \zeta_j]).$$

Then Eqs. (3) and (8) are equivalent to $B(\nu) + \eta^{\mathrm{t}}C(\nu) = 0$ and so $C(\nu)$ gives the matrix of the map $L(\nu)$ in Remark 2.5. The matrix of the transverse Poisson structure is

$$\Lambda_{N,\mu+\nu} = A(\nu) + B(\nu)C(\nu)^{-1}B(\nu)^{\mathrm{t}}$$

and so is polynomial if $C(\nu)$ has a polynomial inverse.

3. Semisimple Lie algebras

This section is devoted to the proof of the following result.

Theorem 3.1. *The Poisson structure Λ_N on an affine subspace transverse to a coadjoint orbit of a semisimple real Lie algebra is a polynomial function of the coordinates.*

Because the Killing form on $\mathfrak{g}$ is nondegenerate, we may identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ and the coadjoint action on $\mathfrak{g}^*$ with the adjoint action on $\mathfrak{g}$.

We first show that we may restrict to orbits through nilpotent elements $\mu = N \in \mathfrak{g}$. Because $\mathfrak{g}$ is a linear Lie algebra, every $A \in \mathfrak{g}$ may be written as $A = S + N$, where S is semisimple, N is nilpotent, and $[S, N] = 0$. By a generalized version of the Jacobson–Morosov theorem, there are elements M, T of $\mathfrak{g}$ which commute with S, and are such that the set $\{N, M, T\}$ spans a subalgebra of $\mathfrak{g}$ which is isomorphic to sl_2. Because N, M, and T commute with S, they lie in the centralizer Z_S of S in $\mathfrak{g}$, which is a real reductive linear Lie subalgebra of $\mathfrak{g}$. In other words, Z_S is the sum of an abelian and a semisimple subalgebra of $\mathfrak{g}$. Because every finite dimensional representation of sl_2 on a real reductive Lie algebra $\mathfrak{H}$ is completely reducible, we may repeat the argument used in the case of a semisimple Lie algebra for the nilpotent $N \in \mathfrak{h} = Z_S$.

Now assume $\mu = N$ is nilpotent. By the Jacobson–Morosov theorem [4], there are $M, T \in \mathfrak{g}$ such that $\{N, M, T\}$ span a subalgebra of $\mathfrak{g}$ which is isomorphic to sl_2, that is

$$[T, N] = 2N, \quad [T, M] = -2M, \quad \text{and} \quad [N, M] = T. \tag{10}$$

The adjoint action of $\mathfrak{g}$ on itself restricts to an action of sl_2 on $\mathfrak{g}$. We recall some basic facts about the finite dimensional representation theory of sl_2, which will be used below. For more background see [6].

From the complete reducibility of finite dimensional representations of sl_2, it follows that

$$\mathfrak{g} = \ker \mathrm{ad}_N \oplus \mathrm{im}\, \mathrm{ad}_M \tag{11}$$

and

$$\mathfrak{g} = \ker \mathrm{ad}_M \oplus \mathrm{im}\, \mathrm{ad}_N. \tag{12}$$

To explain the relation between the above two decompositions, we need more facts from representation theory. Let $\lambda \in \mathbb{R}$ and let

$$V_\lambda = \{v \in \mathfrak{g} \mid \mathrm{ad}_T v = \lambda v\}$$

be the eigenspace of ad_T corresponding to the eigenvalue λ. If $V_\lambda \neq \{0\}$, then λ is a *weight* of the adjoint representation of sl_2 on $\mathfrak{g}$ and V_λ is the corresponding *weight space*. Let $\mathcal{I}$ be the set of all weights. Then $\mathcal{I}$ is a finite subset of $\mathbb{Z}$, which is symmetric about 0, that is, is invariant under multiplication by -1. The weight spaces decompose $\mathfrak{g}$, that is,

$$\mathfrak{g} = \sum_{\lambda \in \mathcal{I}} \oplus V_\lambda. \tag{13}$$

Thus using (13) we may write every $\xi \in \mathfrak{g}$ as

$$\xi = \sum_{\lambda \in \mathcal{I}} \xi_\lambda,$$

where $\xi_\lambda \in V_\lambda$. We say that ξ_λ is the *component* of ξ *with weight* λ. For every λ, $\lambda' \in \mathcal{I}'$ we have

$$[V_\lambda, V_{\lambda'}] \subseteq V_{\lambda+\lambda'}. \tag{14}$$

In (11) $\ker \mathrm{ad}_N$ is the subspace of $\mathfrak{g}$ spanned by the top weight vectors in each irreducible summand of the sl_2 representation, while $\ker \mathrm{ad}_M$ is the subspace spanned by the bottom weight vectors. There is a nonidentity bijective real linear map $\delta : \mathfrak{g} \to \mathfrak{g}$, which for every weight λ maps the weight space V_λ bijectively onto the weight space $V_{-\lambda}$ such that $\delta \circ \delta = \mathrm{id}$. The map δ is the only involution of $\mathfrak{g}$ which commutes with ad_T and intertwines ad_N with ad_M and ad_M with ad_N. It is easy to see that the decomposition (12) is the image of the decomposition (11) under the duality map δ.

Lemma 3.2. *The subspace* $\mathrm{im}\,\mathrm{ad}_M$ *is* ad_T*-invariant.*

Proof. Suppose that $\tau \in \mathrm{im}\,\mathrm{ad}_M$. Then write $\tau = \mathrm{im}\,\mathrm{ad}_M \zeta$ for some $\zeta \in \mathfrak{g}$. Since $[T, M] = -2M$ implies that $[\mathrm{ad}_T, \mathrm{ad}_M] = -2\,\mathrm{ad}_M$, we obtain

$$\mathrm{ad}_T \tau = \mathrm{ad}_T \circ \mathrm{ad}_M \zeta = \mathrm{ad}_M (\mathrm{ad}_T - 2\,\mathrm{id})\zeta,$$

which lies in $\mathrm{im}\,\mathrm{ad}_M$. $\square$

Thus the weight space decomposition of $\mathfrak{g}$ (13) gives rise to a *weight grading* of $\mathrm{im}\,\mathrm{ad}_M$, namely,

$$\mathrm{im}\,\mathrm{ad}_M = \sum_{\lambda \in \mathcal{I}} \oplus (V_\lambda \cap \mathrm{im}\,\mathrm{ad}_M) = \sum_{\lambda \in \mathcal{I}'} \oplus W_\lambda,$$

where $\mathcal{I}'$ is the set of $\lambda \in \mathcal{I}$ such that $W_\lambda \neq \{0\}$. For $\lambda \in \mathcal{I}$ let

$$\mathcal{W}_\lambda = \sum_{\substack{\mu \leqslant \lambda \\ \mu \in \mathcal{I}'}} \oplus W_\mu.$$

Corresponding to the linear ordering $\leqslant$ on $\mathcal{I}'$ is a linear ordering $\subseteq$ on the set of subspaces $\{\mathcal{W}_\lambda\}_{\lambda \in \mathcal{I}'}$. The sets $\mathcal{W}_\lambda$ for $\lambda \in \mathcal{I}'$ define a *filtration* of $\mathrm{im}\,\mathrm{ad}_M$ associated to the weight grading. We set $\mathcal{W}_\mu = \{0\}$ if $\mu < \lambda$ for every $\lambda \in \mathcal{I}'$.

Remark 3.3. The transverse slice and weight grading described above is also used by Slodowy [10] to prove Brieskorn's result that the sets of nilpotent elements of simple Lie algebras have transverse simple singularities at subregular points [1].

We now use the sl_2 representation theory described above to show that the operator $D^{-1}F\nu$ defined in Remark 2.5 is nilpotent for each ν in $\mathfrak{n}_\mu^\circ$ and so the inverse $L(\nu)^{-1}$ is polynomial in ν. The decompositions (11) and (12) show that we can identify

$$\begin{aligned} \mathfrak{g}_\mu &= \ker \mathrm{ad}_N, & \mathfrak{n}_\mu &= \mathrm{im}\,\mathrm{ad}_M, \\ \mathfrak{g}_\mu^\circ &= \mathrm{im}\,\mathrm{ad}_N, & \mathfrak{n}_\mu^\circ &= \ker \mathrm{ad}_M. \end{aligned}$$

The linear maps D and $F\nu$ are defined from $\mathrm{im}\,\mathrm{ad}_M$ to $\mathrm{im}\,\mathrm{ad}_N$ and given by

$$D = -\,\mathrm{ad}_N |_{\mathrm{im}\,\mathrm{ad}_M},$$

$$F\nu = -\pi_2 \circ \mathrm{ad}_\nu |_{\mathrm{im}\,\mathrm{ad}_M},$$

where π_2 be the projection from $\mathfrak{g}$ to $\mathrm{im}\,\mathrm{ad}_N$ with kernel $\ker \mathrm{ad}_M$.

Proposition 3.4. *The composition* $\mathcal{N} = D^{-1}F\nu : \mathrm{im}\,\mathrm{ad}_M \to \mathrm{im}\,\mathrm{ad}_M$ *satisfies* $\mathcal{N}(\mathcal{W}_\lambda) \subseteq \mathcal{W}_{\lambda-2}$ *and is therefore nilpotent.*

Proof. Since ad_N increases weights by 2, the map D increases weights by precisely 2. The space $\ker \mathrm{ad}_M$ consists of bottom weight vectors and so for $\nu \in \ker \mathrm{ad}_M$ the map

ad_ν decreases weights. Since π_2 preserves weights it follows that $F\nu$ must also decrease weights. Hence the composition $D^{-1}F\nu$ decreases weights by at least 2. □

Theorem 3.1 follows from this and Remark 2.5.

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